HOW THE MULTI-VARIABLE CHAIN RULE (CASE 1 VERSION) FOLLOWS FROM THE DIFFERENTIAL dz

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Given a function $z = f(x, y)$ of two variables where $x = x(t)$ and $y = y(t)$ are also functions of t, a change in t value, say by Δt, produces a change Δx in x-value and a change Δy in y-value, which ultimately causes a change Δz in z-value.

The Total Differential Principle says that

$$\Delta z \approx dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy.$$

In any application, $dx = \Delta x$ and $dy = \Delta y$.

So,

$$\Delta z \approx dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y.$$

Now,

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{1}{\Delta t}\right) \cdot \Delta z.$$

using the differential approximation of Δz by dz, these limits then are equal:

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \left(\frac{1}{\Delta t}\right) \cdot \Delta z = \lim_{\Delta t \to 0} \left(\frac{1}{\Delta t}\right) \left(\frac{\partial z}{\partial x} \cdot \Delta x + \frac{\partial z}{\partial y} \cdot \Delta y\right)$$

So,

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \left(\frac{\partial z}{\partial x} \cdot \frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y} \cdot \frac{\Delta y}{\Delta t}\right)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \left(\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}\right) + \frac{\partial z}{\partial y} \left(\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}\right)$$

$$\underline{\underline{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}}$$